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The Lie algebraic structure of symmetries generated by hereditary symmetries†

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Abstract. Based on works of Li, Zhu and co-workers, a general Lie algebraic structure is given for symmetries of integrable systems which are generated by hereditary symmetries.

Chen *et al* (1983) proved that the KP equation possesses two sets of symmetries $\{K_m\}$ and $\{\tau_m\}$ for which it holds that

$$[\bar{K}_m, \bar{K}_n]=0 \qquad [\bar{K}_m, \bar{\tau}_n]=n\bar{K}_{m+n} \qquad [\bar{\tau}_m, \bar{\tau}_n]=(m-n)\bar{\tau}_{m+n}$$

where $\bar{K}_m = K_{m-1}$, $\tau_n = 3\bar{\tau}_{n+2}$. The same algebraic structure of symmetries of the AKNS hierarchy was also exhibited by Li and Zhu (1986). The basic tool in their derivation is the hereditary symmetry which generates the hierarchy. This argument was used by Li and Zhu (1985), Zhu and Li (1985, 1986), Cheng and Li (1987) and Li and Hu (1986) to find similar Lie algebras for a number of integrable systems such as the KdV, KN, Levi, Ito, Boiti-Tu and Tu hierarchies. Tian (1987) found a more general Lie algebraic structure of symmetries of the Burgers equation. Recently Chen *et al* (1987) used the same method once again in their discussion on symmetries of the matrix AKNS hierarchy.

The aim of the present paper is to give a general discussion on symmetries generated by hereditary symmetries. By using the Lie derivatives we show that those symmetries admit an infinite-dimensional Lie algebraic structure which can be identified with the semidirect product of the Kac-Moody and Virasoro algebras.

We recall first some basic notions. Let $\mathbb{R}$ and $\mathbb{C}$ be respectively the real and complex fields, and $\mathscr{S}$ be a topological linear space over $\mathbb{C}$. The differentiable functions mapping $\mathbb{R}\times\mathbb{R}\times\mathscr{S}$ into $\mathscr{S}$ constitute a Lie algebra $\mathscr{L}$ with respect to the Lie product

$$[K(u), S(u)]=(\mathrm{d}/\mathrm{d}\varepsilon)|_{\varepsilon=0}(K(u+\varepsilon S(u))-S(u+\varepsilon K(u))).$$

Let $u=u(x, t)$ be a differentiable function mapping $\mathbb{R}\times\mathbb{R}$ into $\mathscr{S}$. If the equation

$$u_t=K(x, t, u) \qquad K\in\mathscr{L} \tag{1}$$

remains invariant under the infinitesimal transformation $u\to u+\varepsilon G(x, t, u)$, $G\in\mathscr{L}$, or equivalently (Fuchssteiner 1983)

$$G_t=[K, G] \tag{2}$$

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then G is called a symmetry of (1). We denote by $\mathscr{U}$ the set of linear operators mapping $\mathscr{L}$ into itself. Let $\Phi \in \mathscr{U}$ be an opertor, then the Lie deriative (Magri 1980, Tu 1986) Φ_K of Φ along K is defined by

$$(\Phi_K)S \equiv (\mathscr{L}_K \Phi)S = [K, \Phi S] - \Phi[K, S] \qquad S \in \mathscr{L}. \tag{3}$$

An operator Φ is called a hereditary symmetry (Fuchssteiner 1979) if it holds that

$$\Phi^2[K, S] + [\Phi K, \Phi S] = \Phi([\Phi K, S] + [K, \Phi S]) \tag{4}$$

for $K, S \in \mathscr{L}$. It is easy to verify that (4) can be deduced from the equation

$$\Phi'[\Phi K]S - \Phi'[\Phi S]K = \Phi(\Phi'[K]S - \Phi'[S]K)$$

where

$$\Phi'[K]S = (\mathrm{d}/\mathrm{d}\varepsilon)|_{\varepsilon=0}\Phi(u + \varepsilon K)S.$$

From (3) and (4) we deduce immediately the following proposition.

Proposition 1. If Φ is a hereditary symmetry, then

$$\mathscr{L}_{\Phi K}\Phi = \Phi \mathscr{L}_K \Phi \qquad K \in \mathscr{L}. \tag{5}$$

We have also the following proposition.

Proposition 2. If Φ_K is commutative with Φ, i.e. if $(\mathscr{L}_K\Phi)\Phi = \Phi(\mathscr{L}_K\Phi)$, then

$$\mathscr{L}_K(\Phi^n) = n\Phi^{n-1}(\mathscr{L}_K\Phi) \qquad n \geq 0. \tag{6}$$

Proof. By the Leibniz rule of Lie derivatives

$$\mathscr{L}_K(AB) = (\mathscr{L}_K A)B + A(\mathscr{L}_K B)$$

which can be verified directly, we see that

$$\mathscr{L}_K(\Phi^n) = \sum_{i=1}^{n} \Phi^{i-1}(\mathscr{L}_K\Phi)\Phi^{n-i}$$

which, combining the commutative assumption, implies the desired conclusion.

Note that (6) can be written in the form

$$[K, \Phi^n S] = \Phi^n[K, S] + n\Phi^{n-1}\Phi_K S. \tag{7}$$

Proposition 3. If Φ is a hereditary symmetry and $\mathscr{L}_K\Phi$ is commutative with Φ, then

$$\mathscr{L}_{\Phi^m K}(\Phi^n) = n\Phi^{n+m-1}(\Phi_K) \qquad n, m \geq 0. \tag{8}$$

Proof. The case $n = 0$ is trivial since $\mathscr{L}_K I = 0$ for the identity operator I and any $K \in \mathscr{L}$. In the case $n \geq 1$, (8) is an immediate consequence of (5) and (6).

Equation (8) can be written in the form

$$[\Phi^m K, \Phi^n S] = \Phi^n[\Phi^m K, S] + n\Phi^{m+n-1}(\Phi_K S). \tag{9}$$

Proposition 4. If Φ is a hereditary symmetry and if both Φ_K and Φ_S are commutative with Φ, then

$$[\Phi^m K, \Phi^n S] = \Phi^{m+n}[K, S] + \Phi^{m+n-1}(n\Phi_K S - m\Phi_S K).$$

Proof. By the assumption that Φ_S commutes with Φ, we have from (7) that $[S, \Phi^m K] = \Phi^m[S, K] + m\Phi^{m-1}\Phi_S K$. Substituting this equation into (9) we obtain the desired conclusion.

Proposition 5. This is a special case of proposition 4. If $\Phi_K = -\Sigma\, \beta_i \Phi^i \equiv -\beta(\Phi)$ and $\Phi_S = -\Sigma\, \alpha_i \Phi^i = -\alpha(\Phi)$ are polynomials of Φ, and Φ is a hereditary symmetry, then

$$[\Phi^m K, \Phi^n S] = \Phi^{m+n}[K, S] + \Phi^{m+n-1}(m\alpha(\Phi)K - n\beta(\Phi)S).$$

Proposition 6. This is a special case of proposition 5. Let Φ be a hereditary symmetry and suppose that

$$\Phi_K = 0 \qquad \Phi_S = -\alpha \qquad \Phi[K, S] = \beta K$$

where $K, S \in \mathscr{L}$, $\alpha = \alpha(\Phi)$ and $\beta = \beta(\Phi)$. Then

$$[\Phi^m K, \Phi^n K] = 0$$

$$[\Phi^m K, \Phi^n S] = (\alpha m + \beta)\Phi^{m+n-1} K$$

$$[\Phi^m S, \Phi^n S] = \alpha(m - n)\Phi^{m+n-1} S.$$

We shall make use of this proposition in the following to derive the Lie algebraic structures of various integrable evolution equations. To this end we adopt the following definition (Fuchssteiner 1983). Let $K, S \in \mathscr{L}$, if it holds that $\hat{K}^{n+1} S = 0$, where $\hat{K} = \operatorname{ad} K$, i.e. $\hat{K}S = [K, S]$, then S is called a generator of degree n. In this case the expression

$$\sum_{k=0}^{n} (t^k/k!)\hat{K}^k S$$

satisfies (2) and accordingly it represents a symmetry, which depends explicitly on t, of (1). Taking $n = 1$ in particular we obtain the following proposition.

Proposition 7. If $[K, [K, S]] = 0$, then

$$\tau = t[K, S] + S \tag{10}$$

is a symmetry of (1).

We are now in a position to establish the following theorem.

Theorem 1. Let $\Phi = \Phi(u)$, $S = S(u)$ and $T = T(u)$ be not explicitly dependent on t, where $\Phi \in \mathscr{U}$, $S, T \in \mathscr{L}$, and let $K = \Phi T$. Suppose that:

(i) Φ is a hereditary symmetry and $\Phi_K = 0$,

(ii) $\Phi_S = -\alpha$, $\Phi[T, S] = \beta T$, $(\alpha = \alpha(\Phi), \beta = \beta(\Phi))$.

Then the elements

$$K_m = \Phi^m K \qquad \tau_n = \Phi^{n+1}\tau \qquad m \geqslant 0,\ n \geqslant -1$$

where τ is given by (10), form two sets of symmetries of the equation $u_t = K(u)$, and they constitute an infinite-dimensional Lie algebra with the following commutator relations:

$$[K_m, K_n] = 0, \tag{11}$$

$$[K_m, \tau_n] = (\alpha m + \beta_1)K_{m+n} \qquad \beta_1 = \alpha + \beta \tag{12}$$

$$[\tau_m, \tau_n] = \alpha(m - n)\tau_{m+n}. \tag{13}$$

Remark 1. The expressions (12) and (13) are concise forms of the following ones:

$$[K_m, \tau_n] = (m+1)\sum \alpha_i K_{m+n+i} + \sum \beta_j K_{m+n+j}$$

$$[\tau_m, \tau_n] = (m-n)\sum \alpha_i \tau_{m+n+i}$$

which come from (12) and (13) upon the substitution of $\alpha(\Phi) = \Sigma\, \alpha_i \Phi^i$ and $\beta(\Phi) = \Sigma\, \beta_i \Phi^i$.

Remark 2. The condition $\Phi_K = 0$ mentioned in the theorem is equivalent to

$$[K, \Phi S] = \Phi[K, S]. \tag{14}$$

An operator $\Phi \in \mathcal{U}$ which meets (14) is called (Fuchssteiner 1979) a strong symmetry of the equation $u_t = K(u)$.

Proof of theorem 1. By the hypothesis we know that proposition 6 holds with respect to the quadruple (Φ, T, S, β). Hence it holds in particular that $[\Phi T, T] = 0$, $[\Phi T, S] = (\alpha+\beta)T$ and $[\Phi T, \Phi S] = (\alpha+\beta)\Phi T$, or equivalently that

$$[K, T] = 0 \qquad [K, S] = (\alpha+\beta)T \qquad [K, \Phi S] = (\alpha+\beta)K. \tag{15}$$

By (14) we obtain then $\Phi[K, S] = (\alpha+\beta)K = \beta_1 K$. This fact implies that proposition 6 holds again with respect to the quadruple (Φ, K, T, β_1), and consequently we have

$$[\Phi^m K, \Phi^n K] = 0 \tag{16}$$

$$[\Phi^m K, \Phi^n S] = (\alpha m + \beta_1)\Phi^{m+n-1} K \tag{17}$$

$$[\Phi^m S, \Phi^n S] = \alpha(m-n)\Phi^{m+n-1} S. \tag{18}$$

In particular, $[K_m, K] = 0$. Therefore $\{K_m\}$ is a set of symmetries, which are not explicitly dependent on t, of the equation $u_t = K(u)$. Besides, we see from (14) and (15) that

$$[K, [K, \Phi^{n+1} S]] = \Phi^{n+1}[K, [K, S]] = \Phi^{n+1}((\alpha+\beta)[K, T]) = 0. \tag{19}$$

Since by definition and (14) we have

$$\tau_n = \Phi^{n+1}(t[K, S] + S) = t[K, \Phi^{n+1} S] + \Phi^{n+1} S. \tag{20}$$

Equation (19) and proposition 7 imply that $\{\tau_n\}$ constitutes another set of symmetries which are explicitly dependent on t. By the way, we observe that

$$\tau_n = t\beta_1 K_n + \Phi^{n+1} S \tag{21}$$

and by (16) and (17) we deduce

$$[K_m, \tau_n] = [\Phi^m K, \Phi^{n+1} S] = (\alpha m + \beta_1)\Phi^{m+n} K$$

which proves (12). Furthermore we have, by (21) and (16)-(18), that

$$\begin{aligned}
[\tau_m, \tau_n] &= [t\beta_1 K_m + \Phi^{m+1} S, t\beta_1 K_n + \Phi^{n+1} S] \\
&= t\beta_1([K_m, \Phi^{n+1} S] - [K_n, \Phi^{m+1} S]) + [\Phi^{m+1} S, \Phi^{n+1} S] \\
&= t\beta_1((\alpha m + \beta_1)\Phi^{m+n} K - (\alpha n + \beta_1)\Phi^{m+n} K) + \alpha(m-n)\Phi^{m+n-1} S \\
&= \alpha(m-n)(t\beta_1 K_{m+n} + \Phi^{m+n+1} S) = \alpha(m-n)\tau_{m+n}
\end{aligned}$$

which implies (13). Since (11) is the same as (16), the proof of theorem 1 is completed.

The Lie algebra presented in theorem 1 covers those contained in works of Li and Zhu (1985, 1986), Cheng and Li (1987), Li and Hu (1986) Tian (1987) and Chen *et al* (1987). In the special case when $\alpha=\alpha_0=\text{constant}\neq 0$, $\beta=\beta_0=\text{constant}$, we set $\bar{\tau}_n=\tau_n/\alpha$, and $\gamma=\beta/\alpha$, which corresponds to the substitution of Φ by Φ/α. Then the Lie algebraic relations (11)-(13) are as follows:

$$[K_m, K_n]=0$$

$$[K_m, \bar{\tau}_n]=(m+\gamma)K_{m+n}$$

$$[\bar{\tau}_m, \bar{\tau}_n]=(m-n)\bar{\tau}_{m+n}.$$

We have shown (Tu 1987) that this is the semidirect product algebra of the affine Lie algebra $A_1^{(1)}$ and the Virasoro algebra (both without centre). The same conclusion applies also to the general case (11)-(13).

Remark 3. The above τ symmetries τ_n can be replaced by

$$\tau_n^p=\Phi^{n+1}\tau^p \qquad \tau^p\equiv t[K_p, S]+S.$$

Then

$$\tau^p=t(\alpha p+\beta_1)K_{p-1}+S \qquad \tau_n^p=t(\alpha p+\beta_1)K_{n+1}+\Phi^{n+1}S.$$

It is easy to prove that $\{\tau_n^p\}$ and $\{K_m\}$ (with fixed p) constitute two sets of symmetries of the equation

$$u_t=K_p(u)=\Phi^pK(u)$$

and their Lie algebraic structure is the same as given by (11)-(13).

Remark 4. It has long been conjectured that the operator Φ, appearing in a hierarchy of equations $u_t=\Phi^m f(u)$ derived from an isospectral problem, is always a hereditary symmetry. No example has been found yet contrary to this conjecture. Therefore the verification of the hypothesis (i) of theorem 1, though requiring a heavy calculation, presents no essential difficulty, and accordingly the key point in the derivation of Lie algebras is to find a pair of T and S such that

$$\Phi_S=-\alpha \qquad \text{and} \qquad \Phi[T, S]=\beta T.$$

Note that the first of these equations can be replaced by the condition

$$\Phi'[S]=S'\Phi-\Phi S'+\alpha. \tag{22}$$

Example 1. AKNS hierarchy (Li and Zhu 1986). The AKNS hierarchy is

$$u_t=\Phi^p u_x \qquad (p=0,1,2,\ldots)$$

$$u=\begin{bmatrix} q \\ r \end{bmatrix} \qquad \Phi=\mathrm{i}\begin{bmatrix} \partial-2q\partial^{-1}r & -2q\partial^{-1}q \\ 2r\partial^{-1}r & -\partial+2r\partial^{-1}q \end{bmatrix} \qquad \partial=\frac{\mathrm{d}}{\mathrm{d}x}.$$

It is known that Φ is a hereditary symmetry and moreover Φ is a strong symmetry of the equation

$$u_t=K(u)=\Phi u_x. \tag{23}$$

By taking

$$S=\mathrm{i}x\begin{bmatrix} -q \\ r \end{bmatrix} \qquad T=u_x$$

we can easily verify that

$$\Phi_S = 1 \qquad \text{and} \qquad \Phi[T, S] = \Phi\left(\mathrm{i}\begin{bmatrix} -q \\ r \end{bmatrix}\right) = T.$$

Thus we have $\alpha = \beta = 1$ and, by theorem 1, (23) possesses two sets of symmetries

$$K_m = \Phi^m K = \Phi^{m+1} u_x \qquad \tau_n = \Phi^{n+1}\tau \qquad \tau = t[K, S] + S$$

with the Lie algebraic relations

$$[K_m, K_n] = 0$$

$$[K_m, \tau_n] = (m+2) K_{m+n}$$

$$[\tau_m, \tau_n] = (m-n)\tau_{m+n}.$$

The same relations hold also for the equation $u_t = \Phi^p K = K_p$ with $\tau = t[K_p, S] + S$.

Example 2. KdV hierarchy (Li and Zhu 1985). The KdV hierarchy is

$$u_t = \Phi^{p+1} u_x = \Phi^p K \qquad K = \Phi u_x$$

with

$$\Phi = \partial^2 + 4u + 2u_x \partial^{-1} \qquad (\partial = \mathrm{d}/\mathrm{d}x;\ \partial^{-1} 1 = x).$$

Let $S = \frac{1}{2}$, $T = u_x$, then it can be easily verified that

$$\Phi'[S] = S'\Phi - \Phi S' + 2$$

and

$$\Phi[T, S] = (\Phi\partial) S = (2ux)(\tfrac{1}{2}) = T.$$

Therefore we have in this case $\alpha = 2$ and $\beta = 1$, and theorem 1 leads to the following Lie algebra:

$$[K_m, K_n] = 0$$

$$[K_m, \tau_n] = (2m+3) K_{m+n}$$

$$[\tau_m, \tau_n] = 2(m-n)\tau_{m+n}$$

for

$$K_m = \Phi^m K \qquad \tau_n = \Phi^{n+1}\tau \qquad \tau = t[K_p, S] + S$$

which are symmetries of the equations $u_t = \Phi^p K$.

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